The absolute gravity force equation in geodesic notation

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INTRODUCTION

Earlier work gives the force equation for absolute gravity in two different notations: in terms of Christoffel symbols of the first kind[2], and in terms of three-dimensional vector calculus[1].

The main result of this technical note is the force equation in another notation, using Christoffel symbols of the second kind:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -\left(\Gamma^i_{00} + 2\Gamma^i_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right) + \frac{\mathrm{d}x^i}{\mathrm{d}x^0}\left(\Gamma^0_{00} + 2\Gamma^0_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0} + \Gamma^0_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right). \tag{1}$$

This is more similar to the notation used in most textbooks when presenting the geodesic equation.

DERIVATION

We can start from the force equation using Christoffel symbols of the first kind[2]:

$$\frac{\mathrm{d}^2r^i}{\mathrm{d}t^2} = -\left(g^{i\sigma} - \frac{1}{c}g^{0\sigma}\frac{\mathrm{d}r^i}{\mathrm{d}t}\right)[\mu\nu,\sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}t}\frac{\mathrm{d}r^\nu}{\mathrm{d}t} \tag{2}$$

Change notation from r^α to x^α:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -\left(g^{i\sigma} - \frac{1}{c}g^{0\sigma}\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)[\mu\nu,\sigma]\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} \tag{3}$$

Since $x^0 = ct$, change notation from $\mathrm{d}t$ to $\frac{\mathrm{d}x^0}{c}$ and from $\mathrm{d}t^2$ to $\frac{\mathrm{d}(x^0)^2}{c^2}$:

$$c^2\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -\left(g^{i\sigma} - \frac{c}{c}g^{0\sigma}\frac{\mathrm{d}x^i}{\mathrm{d}x^0}\right)[\mu\nu,\sigma]c^2\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0} \tag{4}$$

Cancel factors of c:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -\left(g^{i\sigma} - g^{0\sigma}\frac{\mathrm{d}x^i}{\mathrm{d}x^0}\right)[\mu\nu,\sigma]\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0} \tag{5}$$

Expand:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -g^{i\sigma}[\mu\nu,\sigma]\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0} + \frac{\mathrm{d}x^i}{\mathrm{d}x^0}g^{0\sigma}[\mu\nu,\sigma]\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0} \tag{6}$$

Convert Christoffel symbols of the first kind to the second kind:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -\Gamma^i_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0} + \frac{\mathrm{d}x^i}{\mathrm{d}x^0}\Gamma^0_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0} \tag{7}$$

Expand over μ and ν to separate the cases $\mu = 0$ or $\nu = 0$ from the cases $\mu, \nu = 1, 2, 3$, and then replace μ and ν by j and k for $j, k = 1, 2, 3$:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -\left(\Gamma^i_{00}\frac{\mathrm{d}x^0}{\mathrm{d}x^0}\frac{\mathrm{d}x^0}{\mathrm{d}x^0} + \Gamma^i_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^0}{\mathrm{d}x^0} + \Gamma^i_{0k}\frac{\mathrm{d}x^0}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right) + \frac{\mathrm{d}x^i}{\mathrm{d}x^0}\left(\Gamma^0_{00}\frac{\mathrm{d}x^0}{\mathrm{d}x^0}\frac{\mathrm{d}x^0}{\mathrm{d}x^0} + \Gamma^0_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^0}{\mathrm{d}x^0} + \Gamma^0_{\mu\nu}\frac{\mathrm{d}x^0}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0} + \Gamma^0_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right) \tag{8}$$

$\frac{\mathrm{d}x^0}{\mathrm{d}x^0} = 1$:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -\left(\Gamma^i_{00} + \Gamma^i_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0} + \Gamma^i_{0k}\frac{\mathrm{d}x^k}{\mathrm{d}x^0} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right) + \frac{\mathrm{d}x^i}{\mathrm{d}x^0}\left(\Gamma^0_{00} + \Gamma^0_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0} + \Gamma^0_{\mu\nu}\frac{\mathrm{d}x^k}{\mathrm{d}x^0} + \Gamma^0_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right) \tag{9}$$

Use the symmetry $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$ to coalesce some terms:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}(x^0)^2} = -\left(\Gamma^i_{00} + 2\Gamma^i_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right) + \frac{\mathrm{d}x^i}{\mathrm{d}x^0}\left(\Gamma^0_{00} + 2\Gamma^0_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}x^0} + \Gamma^0_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}x^0}\frac{\mathrm{d}x^k}{\mathrm{d}x^0}\right) \tag{10}$$

This is equation (1).

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[1] Parker, D. B., "The absolute gravity force equation as classical mechanics", 2023, preprint, https://pgu.org
[2] Parker, D. B., "General Relativity in Absolute Space and Time", 2022, preprint, https://pgu.org