Implicit coordinate system transforms

David B. Parker*
pgu.org
(Dated: January 10, 2023)

INTRODUCTION

This technical note describes an algorithm to convert between cartesian coordinates (x, y, z) and another arbitrary coordinate system (u, v, w), when the relationships between the coordinate systems are given as implicit equations. You can derive this algorithm based on information scattered throughout almost every vector calculus textbook, but I find it handy to have the information gathered together in one spot.

This algorithm calculates the differentials $\mathrm{d}x, \ldots, \mathrm{d}w$ and unit vectors $\hat{\mathbf{x}}, \ldots, \hat{\mathbf{w}}$, from the implicit equations. It does not calculate explicit equations for (x, y, z) in terms (u, v, w), or vice-versa, because in the most general case that might be impossible, and is often unenlightening.

ALGORITHM

1. Write the relationships between the (x, y, z) and (u, v, w) as a system of three implicit equations (see Notes 1 and 2):

$$0 = f_1(x, y, z, u, v, w), \tag{1a}$$
$$0 = f_2(x, y, z, u, v, w), \tag{1b}$$
$$0 = f_3(x, y, z, u, v, w). \tag{1c}$$

2. Calculate the partial derivatives of f_1, f_2, and f_3 with respect to x, y, z, u, v, and w, and arrange them in two matrixes $\mathbf{M}$ and $\mathbf{N}$:

$$\mathbf{M} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{bmatrix}. \tag{2}$$

3. Calculate two matrixes $\mathbf{P}$ and $\mathbf{Q}$ from $\mathbf{M}$ and $\mathbf{N}$ (see Note 3):

$$\mathbf{P} = -\mathbf{M}^{-1}\mathbf{N}, \qquad \mathbf{Q} = -\mathbf{N}^{-1}\mathbf{M}. \tag{3}$$

   The elements of $\mathbf{P}$ and $\mathbf{Q}$ are p_{ij} and q_{ij}, where i is the row and j is the column.

4. Calculate three scale factors s_1, s_2, s_3 (see Note 4):

$$s_1 = \sqrt{p_{11}^2 + p_{21}^2 + p_{31}^2}, \quad s_2 = \sqrt{p_{12}^2 + p_{22}^2 + p_{32}^2}, \quad s_3 = \sqrt{p_{13}^2 + p_{23}^2 + p_{33}^2}. \tag{4}$$

5. The differentials and unit vectors are then (see Note 5):

$$\mathrm{d}x = p_{11}\,\mathrm{d}u + p_{12}\,\mathrm{d}v + p_{13}\,\mathrm{d}w, \tag{5a}$$
$$\mathrm{d}y = p_{21}\,\mathrm{d}u + p_{22}\,\mathrm{d}v + p_{23}\,\mathrm{d}w, \tag{5b}$$
$$\mathrm{d}z = p_{31}\,\mathrm{d}u + p_{32}\,\mathrm{d}v + p_{33}\,\mathrm{d}w, \tag{5c}$$
$$\hat{\mathbf{x}} = q_{11}\,s_1\,\hat{\mathbf{u}} + q_{21}\,s_2\,\hat{\mathbf{v}} + q_{31}\,s_3\,\hat{\mathbf{w}}, \tag{5d}$$
$$\hat{\mathbf{y}} = q_{12}\,s_1\,\hat{\mathbf{u}} + q_{22}\,s_2\,\hat{\mathbf{v}} + q_{32}\,s_3\,\hat{\mathbf{w}}, \tag{5e}$$
$$\hat{\mathbf{z}} = q_{13}\,s_1\,\hat{\mathbf{u}} + q_{23}\,s_2\,\hat{\mathbf{v}} + q_{33}\,s_3\,\hat{\mathbf{w}}, \tag{5f}$$

and

$$\mathrm{d}u = q_{11}\,\mathrm{d}x + q_{12}\,\mathrm{d}y + q_{13}\,\mathrm{d}z, \tag{6a}$$
$$\mathrm{d}v = q_{21}\,\mathrm{d}x + q_{22}\,\mathrm{d}y + q_{23}\,\mathrm{d}z, \tag{6b}$$
$$\mathrm{d}w = q_{31}\,\mathrm{d}x + q_{32}\,\mathrm{d}y + q_{33}\,\mathrm{d}z, \tag{6c}$$
$$\hat{\mathbf{u}} = \frac{1}{s_1}(p_{11}\,\hat{\mathbf{x}} + p_{21}\,\hat{\mathbf{y}} + p_{31}\,\hat{\mathbf{z}}), \tag{6d}$$
$$\hat{\mathbf{v}} = \frac{1}{s_2}(p_{12}\,\hat{\mathbf{x}} + p_{22}\,\hat{\mathbf{y}} + p_{32}\,\hat{\mathbf{z}}), \tag{6e}$$
$$\hat{\mathbf{w}} = \frac{1}{s_3}(p_{13}\,\hat{\mathbf{x}} + p_{23}\,\hat{\mathbf{y}} + p_{33}\,\hat{\mathbf{z}}). \tag{6f}$$

6. Optional: if it is important that the (u, v, w) coordinate system be orthonormal, check that the dot products of the different unit vectors are 0:

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0, \qquad \hat{\mathbf{u}} \cdot \hat{\mathbf{w}} = 0, \qquad \hat{\mathbf{v}} \cdot \hat{\mathbf{w}} = 0. \tag{7}$$

NOTES

1. Often you will have explicit equations for the coordinates:

$$x = g_x(u, v, w), \qquad u = h_u(x, y, z), \tag{8a}$$
$$y = g_y(u, v, w), \qquad \text{and/or} \qquad v = h_v(x, y, z), \tag{8b}$$
$$z = g_z(u, v, w), \qquad w = h_w(x, y, z). \tag{8c}$$

   Then you can construct f_1, f_2, and f_3 from either set of explicit equations:

$$f_1 = x - g_x(u, v, w), \qquad f_1 = u - h_u(x, y, z), \tag{9a}$$
$$f_2 = y - g_y(u, v, w), \qquad \text{or} \qquad f_2 = v - h_v(x, y, z), \tag{9b}$$
$$f_3 = z - g_z(u, v, w), \qquad f_3 = w - h_w(x, y, z). \tag{9c}$$

   Using either set of equations for f_1, f_2, and f_3 should give equivalent results. However, one set or the other might give the results in terms that are nicer for your purpose, so it might be good to try both sets if you have them.

2. Sometimes the same coordinate names are used in different coordinate systems. For example, z appears in both cartesian coordinates (x, y, z) and in cylindrical coordinates (r, θ, z). When transforming between such coordinate systems it is important to distinguish between identically named coordinates, even if they have identical definitions. So, for example, you might want to refer to the z in cylindrical coordinates as z_c. Otherwise, you may end up with situations where

$$0 = z - z \qquad \Longrightarrow \qquad 0 = 0, \tag{10}$$

   when what you really want is

$$0 = z - z_c \qquad \Longrightarrow \qquad z = z_c. \tag{11}$$

3. $\mathbf{P}$ and $\mathbf{Q}$ are inverses of each other so you could also calculate $\mathbf{Q}^{-1}$ by inverting $\mathbf{P}$, or vice-versa.

4. Each s is the sum of the squares of each column of $\mathbf{P}$, not of each row of $\mathbf{P}$.

5. The coefficients p_{ij} and q_{ij} for the unit vectors are the transposes of the coefficients for the differentials.

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* Electronic address: `daveparker@pgu.org`