The absolute gravity force equation as classical mechanics

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When general relativity is specialized to absolute space and time, the geodesic equation of general relativity becomes the force equation of absolute gravity. The force equation is three-dimensional in space with absolute time, so this paper shows how to write the force equation in the scalar/vector/matrix notation of three-dimensional classical mechanics. This reduces gravitation to three-dimensional vector calculus, and make it clear that space and time are not curved in absolute gravity.

INTRODUCTION

The force equation for absolute gravity[1] is three-dimensional in absolute space with absolute time, but that isn't very obvious when the force equation is written in the notation of general relativity:

$$\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2} = -\left(g^{i\sigma} - \frac{1}{c} g^{0\sigma} \frac{\mathrm{d}r^i}{\mathrm{d}t}\right) [\mu\nu, \sigma] \frac{\mathrm{d}r^\mu}{\mathrm{d}t} \frac{\mathrm{d}r^\nu}{\mathrm{d}t} \tag{1}$$

The purpose of this paper is to put the force equation of absolute gravity into the standard three-dimensional scalar/vector/matrix notation of classical mechanics. That reduces gravitation to three-dimensional vector calculus, and it clearly shows that absolute gravity involves no curvature of space or time. In addition to the force equation, the field equation for absolute gravity can also be converted to the notation of classical mechanics using the same techniques, but I think that deserves a paper of its own.

General case: the main result of this paper is that if a particle is at absolute position $\mathbf{r} = [x, y, z]^\mathsf{T}$, with absolute velocity $\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, then the absolute acceleration $\mathbf{a} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$ in the three fields g, $\mathbf{w}$, and $\mathbf{S}$ is:

$$\mathbf{a} = -\left(\overline{\mathbf{w}} - \frac{1}{c}\overline{g}\,\mathbf{v}\right) b - \overline{\mathbf{S}}\,\mathbf{d} + \frac{1}{c}(\overline{\mathbf{w}} \cdot \mathbf{d})\mathbf{v}, \tag{2}$$

where

$$g = g_{00}, \qquad \mathbf{w} = \begin{bmatrix} g_{10} \\ g_{20} \\ g_{30} \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \tag{3a}$$

$$b = \frac{c}{2}\frac{\partial g}{\partial t} + c\frac{\partial g}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{1}{2}\left(\nabla_\mathsf{T}\mathbf{w} + \nabla^\mathsf{T}\mathbf{w} - \frac{1}{c}\frac{\partial \mathbf{S}}{\partial t}\right) \cdot (\mathbf{v}\mathbf{v}^\mathsf{T}), \tag{3b}$$

$$\mathbf{d} = c\frac{\partial \mathbf{w}}{\partial t} - \frac{c^2}{2}\frac{\partial g}{\partial \mathbf{r}} + c\left(\frac{1}{c}\frac{\partial \mathbf{S}}{\partial t} + \nabla_\mathsf{T}\mathbf{w} - \nabla^\mathsf{T}\mathbf{w}\right)\mathbf{v} + \frac{1}{2}\left(\nabla_\mathsf{T}\mathbf{S} + \nabla^\mathsf{T}\mathbf{S} - \frac{\partial \mathbf{S}}{\partial \mathbf{r}}\right) \cdot (\mathbf{v}\mathbf{v}^\mathsf{T}), \tag{3c}$$

$$\mathbf{u} = \mathbf{S}^{-1}\mathbf{w}, \qquad \overline{g} = (g - \mathbf{w} \cdot \mathbf{u})^{-1}, \qquad \overline{\mathbf{w}} = -\overline{g}\,\mathbf{u}, \qquad \overline{\mathbf{S}} = \mathbf{S}^{-1} + \overline{g}\,\mathbf{u}\mathbf{u}^\mathsf{T}. \tag{3d}$$

The mass of the particle does not appear in equations (2) through (3d) because the gravitational force is independent of the particle's mass.

The $g_{\mu\nu}$ elements of g, $\mathbf{w}$, and $\mathbf{S}$ in equations (3a) are the corresponding elements of the metric tensor, broken out into three-dimensional scalar/vector/matrix form (see equation (19)). The overbar on the variables $\overline{g}$, $\overline{\mathbf{w}}$, and $\overline{\mathbf{S}}$ indicates that they bear the same relationship to the inverse metric tensor $g^{\mu\nu}$ as g, $\mathbf{w}$, and $\mathbf{S}$ bear to the metric tensor $g_{\mu\nu}$ (see equation (25)). However, directly inverting $g_{\mu\nu}$ to get $g^{\mu\nu}$ is a four-by-four matrix operation, which violates the self-imposed restriction to only three-dimensional operations, so $\overline{g}$, $\overline{\mathbf{w}}$, and $\overline{\mathbf{S}}$ are instead calculated using a sequence of purely three-dimensional operations (see equation (28)).

I made up notations for some of the operations in equations (3b) and (3c), most obviously ∇_T and ∇^T, because I couldn't find any kind of standard notation. I explicitly write out the definitions in the section on "Explicit vector and matrix operations".

Equations (2) through (3d) may appear to be complicated, but they become much simpler in several common and useful circumstances.

Stationary field case (e.g. Kerr metric): The time derivatives of the fields g, $\mathbf{w}$, and $\mathbf{S}$ are 0. Only b and $\mathbf{d}$ become simpler:

$$b = c\frac{\partial g}{\partial \mathbf{r}} \cdot \mathbf{v} + \frac{1}{2}\left(\nabla_{\mathsf{T}}\mathbf{w} + \nabla^{\mathsf{T}}\mathbf{w}\right) \cdot (\mathbf{v}\mathbf{v}^{\mathsf{T}}), \tag{4a}$$

$$\mathbf{d} = -\frac{c^2}{2}\frac{\partial g}{\partial \mathbf{r}} + c\left(\nabla_{\mathsf{T}}\mathbf{w} - \nabla^{\mathsf{T}}\mathbf{w}\right)\mathbf{v} + \frac{1}{2}\left(\nabla_{\mathsf{T}}\mathbf{S} + \nabla^{\mathsf{T}}\mathbf{S} - \frac{\partial \mathbf{S}}{\partial \mathbf{r}}\right) \cdot (\mathbf{v}\mathbf{v}^{\mathsf{T}}). \tag{4b}$$

Stationary field case, zero velocity case (e.g. Kerr metric with a stationary test particle): The time derivatives of the fields g, $\mathbf{w}$, and $\mathbf{S}$ are 0, and the velocity $\mathbf{v}$ is 0. Almost everything in equations (4a) and (4b) goes to 0, leaving:

$$b = 0, \qquad \mathbf{d} = -\frac{c^2}{2}\frac{\partial g}{\partial \mathbf{r}}, \tag{5}$$

resulting in the force equation (2) simplifying to:

$$\mathbf{a} = \frac{c^2}{2}\overline{\mathbf{S}}\,\frac{\partial g}{\partial \mathbf{r}}. \tag{6}$$

Stationary field, zero field momentum case (e.g. Schwarzschild metric, static weak field metric): The time derivatives of the fields g, $\mathbf{w}$, and $\mathbf{S}$ are 0, and the momentum field $\mathbf{w}$ is 0. This results in almost everything in equations (4a) and (4b) going to 0, leaving:

$$b = c\frac{\partial g}{\partial \mathbf{r}} \cdot \mathbf{v}, \tag{7a}$$

$$\mathbf{d} = -\frac{c^2}{2}\frac{\partial g}{\partial \mathbf{r}} + \frac{1}{2}\left(\nabla_{\mathsf{T}}\mathbf{S} + \nabla^{\mathsf{T}}\mathbf{S} - \frac{\partial \mathbf{S}}{\partial \mathbf{r}}\right) \cdot (\mathbf{v}\mathbf{v}^{\mathsf{T}}), \tag{7b}$$

$$\mathbf{u} = \mathbf{0}, \qquad \overline{g} = \frac{1}{g}, \qquad \overline{\mathbf{w}} = \mathbf{0}, \qquad \overline{\mathbf{S}} = \mathbf{S}^{-1}, \tag{7c}$$

resulting in the force equation (2) simplifying to:

$$\mathbf{a} = \frac{b}{cg}\mathbf{v} - \mathbf{S}^{-1}\mathbf{d}. \tag{8}$$

Stationary field, zero field momentum, zero velocity case (e.g. Schwarzchild metric or static weak field metric, with a stationary test particle): The time derivatives of the fields g, $\mathbf{w}$, and $\mathbf{S}$ are 0, the momentum field $\mathbf{w}$ is 0, and the velocity $\mathbf{v}$ is 0. This results in almost everything in equations (7a) through (7c) going to 0, leaving:

$$b = 0, \qquad \mathbf{d} = -\frac{c^2}{2}\frac{\partial g}{\partial \mathbf{r}}, \tag{9}$$

resulting in the force equation (8) simplifying to:

$$\mathbf{a} = \frac{c^2}{2}\mathbf{S}^{-1}\frac{\partial g}{\partial \mathbf{r}}. \tag{10}$$

DERIVATION OF THE GENERAL CASE

The first step is to expand the sums over μ and ν in force equation (1) in order to separate the cases $\mu = 0$ or $\nu = 0$ from the cases $\mu, \nu = 1, 2, 3$, and then replace μ and ν by j and k for $j, k = 1, 2, 3$:

$$\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2} = -\left(g^{i\sigma} - \frac{1}{c}g^{0\sigma}\frac{\mathrm{d}r^i}{\mathrm{d}t}\right)\left([00,\sigma]\frac{\mathrm{d}r^0}{\mathrm{d}t}\frac{\mathrm{d}r^0}{\mathrm{d}t} + 2[j0,\sigma]\frac{\mathrm{d}r^j}{\mathrm{d}t}\frac{\mathrm{d}r^0}{\mathrm{d}t} + [jk,\sigma]\frac{\mathrm{d}r^j}{\mathrm{d}t}\frac{\mathrm{d}r^k}{\mathrm{d}t}\right) \tag{11}$$

Noting that $r^0 = ct$ (see [1]), we can simplify $\frac{\mathrm{d}r^0}{\mathrm{d}t}$ to c:

$$\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2} = -\left(g^{i\sigma} - \frac{1}{c}g^{0\sigma}\frac{\mathrm{d}r^i}{\mathrm{d}t}\right)\left(c^2[00,\sigma] + 2c[j0,\sigma]\frac{\mathrm{d}r^j}{\mathrm{d}t} + [jk,\sigma]\frac{\mathrm{d}r^j}{\mathrm{d}t}\frac{\mathrm{d}r^k}{\mathrm{d}t}\right). \tag{12}$$

We can begin converting to vector notation by substituting $\mathbf{a}_i$ for $\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2}$, and $\mathbf{v}_{i,j,k}$ for $\frac{\mathrm{d}r^{i,j,k}}{\mathrm{d}t}$, giving:

$$\mathbf{a}_i = -\left(g^{i\sigma} - \frac{1}{c}g^{0\sigma}\mathbf{v}_i\right)\left(c^2[00,\sigma] + 2c[j0,\sigma]\mathbf{v}_j + [jk,\sigma]\mathbf{v}_j\mathbf{v}_k\right) \tag{13}$$

We can now expand the sums over $\sigma = 0,1,2,3$ into separate terms for $\sigma = 0$ and $\sigma = 1,2,3$, and then replace σ by l for $l = 1,2,3$:

$$\begin{aligned}\mathbf{a}_i = &-\left(g^{i0} - \frac{1}{c}g^{00}\mathbf{v}_i\right)\left(c^2[00,0] + 2c[j0,0]\mathbf{v}_j + [jk,0]\mathbf{v}_j\mathbf{v}_k\right)\\ &-\left(g^{il} - \frac{1}{c}g^{0l}\mathbf{v}_i\right)\left(c^2[00,l] + 2c[j0,l]\mathbf{v}_j + [jk,l]\mathbf{v}_j\mathbf{v}_k\right).\end{aligned} \tag{14}$$

To simplify a bit, we can introduce a scalar b and a three-vector $\mathbf{d}_l$:

$$b = c^2[00,0] + 2c[j0,0]\mathbf{v}_j + [jk,0]\mathbf{v}_j\mathbf{v}_k, \tag{15}$$
$$\mathbf{d}_l = c^2[00,l] + 2c[j0,l]\mathbf{v}_j + [jk,l]\mathbf{v}_j\mathbf{v}_k, \tag{16}$$

so that equation (14) becomes:

$$\mathbf{a}_i = -\left(g^{i0} - \frac{1}{c}g^{00}\mathbf{v}_i\right)b - \left(g^{il} - \frac{1}{c}g^{0l}\mathbf{v}_i\right)\mathbf{d}_l. \tag{17}$$

At this point, it becomes convenient to substitute for the Christoffel symbols of the first kind in equations (15) and (16) using the standard formula:

$$[\mu\nu,\sigma] = \frac{1}{2}\left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{18}$$

To keep things three-dimensional, we can break up the metric tensor $g_{\mu\nu}$ into a scalar g, a three-vector $\mathbf{w}$, and a three-by-three matrix $\mathbf{S}$: equation

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03}\\ g_{10} & g_{11} & g_{12} & g_{13}\\ g_{20} & g_{21} & g_{22} & g_{23}\\ g_{30} & g_{31} & g_{32} & g_{33}\end{bmatrix} = \begin{bmatrix} g & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\\ \mathbf{w}_1 & \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{S}_{13}\\ \mathbf{w}_2 & \mathbf{S}_{21} & \mathbf{S}_{22} & \mathbf{S}_{23}\\ \mathbf{w}_3 & \mathbf{S}_{31} & \mathbf{S}_{32} & \mathbf{S}_{33}\end{bmatrix} = \begin{bmatrix} g & \mathbf{w}^\mathsf{T}\\ \mathbf{w} & \mathbf{S}\end{bmatrix}, \tag{19}$$

which gives us equations (3a).

Using equations (18) and (19), we can write the Christoffel symbols of the first kind used in (15) and (16) as:

$$[00,0] = \frac{1}{2c}\frac{\partial g}{\partial t}, \qquad [j0,0] = \frac{1}{2}\frac{\partial g}{\partial \mathbf{r}_j}, \qquad [jk,0] = \frac{1}{2}\left(\frac{\partial \mathbf{w}_j}{\partial \mathbf{r}_k} + \frac{\partial \mathbf{w}_k}{\partial \mathbf{r}_j} - \frac{1}{c}\frac{\partial \mathbf{S}_{jk}}{\partial t}\right), \tag{20a}$$
$$[00,l] = \frac{1}{c}\frac{\partial \mathbf{w}_l}{\partial t} - \frac{1}{2}\frac{\partial g}{\partial \mathbf{r}_l}, \quad [j0,l] = \frac{1}{2}\left(\frac{1}{c}\frac{\partial \mathbf{S}_{lj}}{\partial t} + \frac{\partial \mathbf{w}_l}{\partial \mathbf{r}_j} - \frac{\partial \mathbf{w}_j}{\partial \mathbf{r}_l}\right), \quad [jk,l] = \frac{1}{2}\left(\frac{\partial \mathbf{S}_{lj}}{\partial \mathbf{r}_k} + \frac{\partial \mathbf{S}_{kl}}{\partial \mathbf{r}_j} - \frac{\partial \mathbf{S}_{jk}}{\partial \mathbf{r}_l}\right). \tag{20b}$$

Plugging the Christoffel symbols of the first kind from equations (20a) through (20b) into equation (15) for b and equation (16) for $\mathbf{d}$ gives:

$$b = c^2\frac{1}{2c}\frac{\partial g}{\partial t} + 2c\frac{1}{2}\frac{\partial g}{\partial \mathbf{r}_j}\mathbf{v}_j + \frac{1}{2}\left(\frac{\partial \mathbf{w}_j}{\partial \mathbf{r}_k} + \frac{\partial \mathbf{w}_k}{\partial \mathbf{r}_j} - \frac{1}{c}\frac{\partial \mathbf{S}_{jk}}{\partial t}\right)\mathbf{v}_j\mathbf{v}_k, \tag{21}$$
$$\mathbf{d}_l = c^2\left(\frac{1}{c}\frac{\partial \mathbf{w}_l}{\partial t} - \frac{1}{2}\frac{\partial g}{\partial \mathbf{r}_l}\right) + 2c\frac{1}{2}\left(\frac{1}{c}\frac{\partial \mathbf{S}_{lj}}{\partial t} + \frac{\partial \mathbf{w}_l}{\partial \mathbf{r}_j} - \frac{\partial \mathbf{w}_j}{\partial \mathbf{r}_l}\right)\mathbf{v}_j + \frac{1}{2}\left(\frac{\partial \mathbf{S}_{lj}}{\partial \mathbf{r}_k} + \frac{\partial \mathbf{S}_{kl}}{\partial \mathbf{r}_j} - \frac{\partial \mathbf{S}_{jk}}{\partial \mathbf{r}_l}\right)\mathbf{v}_j\mathbf{v}_k. \tag{22}$$

We can now remove all of the j,k,l indexes in equations (21) and (22) by using the vector and matrix operations in the section "Explicit vector and matrix operations", to get:

$$b = \frac{c}{2}\frac{\partial g}{\partial t} + c\frac{\partial g}{\partial \mathbf{r}}\cdot\mathbf{v} + \frac{1}{2}\left(\nabla_\mathsf{T}\mathbf{w} + \nabla^\mathsf{T}\mathbf{w} - \frac{1}{c}\frac{\partial \mathbf{S}}{\partial t}\right)\cdot(\mathbf{v}\mathbf{v}^\mathsf{T}), \tag{23}$$
$$\mathbf{d} = c\frac{\partial \mathbf{w}}{\partial t} - \frac{c^2}{2}\frac{\partial g}{\partial \mathbf{r}} + c\left(\frac{1}{c}\frac{\partial \mathbf{S}}{\partial t} + \nabla_\mathsf{T}\mathbf{w} - \nabla^\mathsf{T}\mathbf{w}\right)\mathbf{v} + \frac{1}{2}\left(\nabla_\mathsf{T}\mathbf{S} + \nabla^\mathsf{T}\mathbf{S} - \frac{\partial \mathbf{S}}{\partial \mathbf{r}}\right)\cdot(\mathbf{v}\mathbf{v}^\mathsf{T}), \tag{24}$$

which are equations (3b) and (3c).

The next task is to remove the elements of the inverse metric tensor $g^{\mu\nu}$ from equation (17). We can write $g^{\mu\nu}$ in terms of a scalar $\overline{g}$, a three-vector $\overline{\mathbf{w}}$, and a three-by-three matrix $\overline{\mathbf{S}}$ (analogous to equation (19)):

$$g^{\mu\nu} = \begin{bmatrix} g^{00} & g^{01} & g^{02} & g^{03} \\ g^{10} & g^{11} & g^{12} & g^{13} \\ g^{20} & g^{21} & g^{22} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} \overline{g} & \overline{\mathbf{w}}_1 & \overline{\mathbf{w}}_2 & \overline{\mathbf{w}}_3 \\ \overline{\mathbf{w}}_1 & \overline{\mathbf{S}}_{11} & \overline{\mathbf{S}}_{12} & \overline{\mathbf{S}}_{13} \\ \overline{\mathbf{w}}_2 & \overline{\mathbf{S}}_{21} & \overline{\mathbf{S}}_{22} & \overline{\mathbf{S}}_{23} \\ \overline{\mathbf{w}}_3 & \overline{\mathbf{S}}_{31} & \overline{\mathbf{S}}_{32} & \overline{\mathbf{S}}_{33} \end{bmatrix} = \begin{bmatrix} \overline{g} & \overline{\mathbf{w}}^\intercal \\ \overline{\mathbf{w}} & \overline{\mathbf{S}} \end{bmatrix}. \tag{25}$$

Plugging the elements from equation (25) into equation (17) gives:

$$\mathbf{a}_i = -\left(\overline{\mathbf{w}}_i - \frac{1}{c}\overline{g}\mathbf{v}_i\right) b - \left(\overline{\mathbf{S}}_{il} - \frac{1}{c}\overline{\mathbf{w}}_l{}^\intercal \mathbf{v}_i\right) \mathbf{d}_l \tag{26}$$

We can use vector and matrix operations to remove the indexes i and l to get:

$$\mathbf{a} = -\left(\overline{\mathbf{w}} - \frac{1}{c}\overline{g}\mathbf{v}\right) b - \overline{\mathbf{S}}\mathbf{d} + \frac{1}{c}(\overline{\mathbf{w}} \cdot \mathbf{d})\mathbf{v}, \tag{27}$$

which is equation (2).

Our last task is to calculate $\overline{g}$, $\overline{\mathbf{w}}$, and $\overline{\mathbf{S}}$, from g, $\mathbf{w}$, and $\mathbf{S}$. We will use the formula for the block matrix inverse from block matrix algebra[2]. Applying the block matrix inverse to equation (19), starting from equation (25) we get:

$$\begin{aligned} \begin{bmatrix} \overline{g} & \overline{\mathbf{w}}^\intercal \\ \overline{\mathbf{w}} & \overline{\mathbf{S}} \end{bmatrix} &= \begin{bmatrix} g & \mathbf{w}^\intercal \\ \mathbf{w} & \mathbf{S} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (g - \mathbf{w}^\intercal\mathbf{S}^{-1}\mathbf{w})^{-1} & -(g - \mathbf{w}^\intercal\mathbf{S}^{-1}\mathbf{w})^{-1}\mathbf{w}^\intercal\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{w}(g - \mathbf{w}^\intercal\mathbf{S}^{-1}\mathbf{w})^{-1} & \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{w}(g - \mathbf{w}^\intercal\mathbf{S}^{-1}\mathbf{w})^{-1}\mathbf{w}^\intercal\mathbf{S}^{-1} \end{bmatrix}. \end{aligned} \tag{28}$$

We can now read off the formulas for $\overline{g}$, $\overline{\mathbf{w}}$, and $\overline{\mathbf{S}}$, from equation (28). To make the formulas simpler, we will take advantage of the fact that S^{-1} is symmetric because $g_{\mu\nu}$ is symmetric, so that $S^{-1\intercal} = S^{-1}$. We will also introduce a three-vector $\mathbf{u}$ to eliminate several common subexpressions, to get the formulas:

$$\mathbf{u} = \mathbf{S}^{-1}\mathbf{w} \tag{29}$$

$$\overline{g} = (g - \mathbf{w} \cdot \mathbf{u})^{-1} \tag{30}$$

$$\overline{\mathbf{w}} = -\overline{g}\,\mathbf{u} \tag{31}$$

$$\overline{\mathbf{S}} = \mathbf{S}^{-1} + \overline{g}\,\mathbf{u}\mathbf{u}^\intercal. \tag{32}$$

These are the final equations (3d) of the general case.

EXPLICIT VECTOR AND MATRIX OPERATIONS

Most of the vector and matrix operations in equations (2) through (3d) are standard. In this section I explicitly write out the definitions that I feel are uncommon, ambiguous, or nonstandard.

Matrix dot matrix = scalar: In equation (3b), the dot $\cdot$ between $\frac{1}{2}\left(\nabla_\intercal \mathbf{w} + \nabla^\intercal \mathbf{w} - \frac{1}{c}\frac{\partial \mathbf{S}}{\partial t}\right)$ and $(\mathbf{v}\mathbf{v}^\intercal)$ indicates a matrix dot product. Given two matrixes $\mathbf{M}$ and $\mathbf{Q}$, the matrix dot product results in a scalar $l_{\mathbf{MQ}}$:

$$\begin{aligned} \mathbf{M} \cdot \mathbf{Q} &= \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \cdot \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} m_{ij}q_{ij} \\ &= m_{11}q_{11} + m_{12}q_{12} + m_{13}q_{13} + m_{21}q_{21} + m_{22}q_{22} + m_{23}q_{23} + m_{31}q_{31} + m_{32}q_{32} + m_{33}q_{33} \\ &= l_{\mathbf{MQ}}. \end{aligned} \tag{33}$$

Vector of matrixes dot matrix = vector of scalars: In equation (3c), the dot $\cdot$ between $\frac{1}{2}\left(\nabla_{\mathsf{T}}\mathbf{S} + \nabla^{\mathsf{T}}\mathbf{S} - \frac{\partial \mathbf{S}}{\partial \mathbf{r}}\right)$ and $(\mathbf{v}\mathbf{v}^{\mathsf{T}})$ is a straightforward extension of equation (33) to the case where one of the operands is a column vector of matrixes (e.g. resulting from equation (37)):

$$\begin{bmatrix} \mathbf{M} \\ \mathbf{N} \\ \mathbf{P} \end{bmatrix} \cdot \mathbf{Q} = \begin{bmatrix} \mathbf{M}\cdot\mathbf{Q} \\ \mathbf{N}\cdot\mathbf{Q} \\ \mathbf{P}\cdot\mathbf{Q} \end{bmatrix} = \begin{bmatrix} l_{\mathbf{MQ}} \\ l_{\mathbf{NQ}} \\ l_{\mathbf{PQ}} \end{bmatrix}. \tag{34}$$

Simple derivatives: I don't think these are at all controversial, but I thought I would include them for completeness:

$$\frac{\partial \mathbf{w}}{\partial t} = \begin{bmatrix} \frac{\partial \mathbf{w}_1}{\partial t} \\ \frac{\partial \mathbf{w}_2}{\partial t} \\ \frac{\partial \mathbf{w}_3}{\partial t} \end{bmatrix}, \quad \frac{\partial \mathbf{S}}{\partial t} = \begin{bmatrix} \frac{\partial \mathbf{S}_{11}}{\partial t} & \frac{\partial \mathbf{S}_{12}}{\partial t} & \frac{\partial \mathbf{S}_{13}}{\partial t} \\ \frac{\partial \mathbf{S}_{21}}{\partial t} & \frac{\partial \mathbf{S}_{22}}{\partial t} & \frac{\partial \mathbf{S}_{23}}{\partial t} \\ \frac{\partial \mathbf{S}_{31}}{\partial t} & \frac{\partial \mathbf{S}_{32}}{\partial t} & \frac{\partial \mathbf{S}_{33}}{\partial t} \end{bmatrix}. \tag{35}$$

Gradients of scalars and vectors: I think that the standard scalar gradient operation ∇g is ambiguous; is the result a column vector or row vector? I think that writing it as $\frac{\partial g}{\partial \mathbf{r}}$ is less ambigous. The gradients of vectors are also ambiguous, and I needed to use both variations (which are transposes of each other), so I used ∇_{T} when the bottom vector $\mathbf{r}$ is transposed, and ∇^{T} when the top vector $\mathbf{w}$ is transposed.

$$\frac{\partial g}{\partial \mathbf{r}} = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{bmatrix}, \quad \nabla_{\mathsf{T}}\mathbf{w} = \frac{\partial \mathbf{w}}{\partial \mathbf{r}^{\mathsf{T}}} = \begin{bmatrix} \frac{\partial \mathbf{w}_1}{\partial x} & \frac{\partial \mathbf{w}_1}{\partial y} & \frac{\partial \mathbf{w}_1}{\partial z} \\ \frac{\partial \mathbf{w}_2}{\partial x} & \frac{\partial \mathbf{w}_2}{\partial y} & \frac{\partial \mathbf{w}_2}{\partial z} \\ \frac{\partial \mathbf{w}_3}{\partial x} & \frac{\partial \mathbf{w}_3}{\partial y} & \frac{\partial \mathbf{w}_3}{\partial z} \end{bmatrix}, \quad \nabla^{\mathsf{T}}\mathbf{w} = \frac{\partial \mathbf{w}^{\mathsf{T}}}{\partial \mathbf{r}} = \begin{bmatrix} \frac{\partial \mathbf{w}_1}{\partial x} & \frac{\partial \mathbf{w}_2}{\partial x} & \frac{\partial \mathbf{w}_3}{\partial x} \\ \frac{\partial \mathbf{w}_1}{\partial y} & \frac{\partial \mathbf{w}_2}{\partial y} & \frac{\partial \mathbf{w}_3}{\partial y} \\ \frac{\partial \mathbf{w}_1}{\partial z} & \frac{\partial \mathbf{w}_2}{\partial z} & \frac{\partial \mathbf{w}_3}{\partial z} \end{bmatrix}. \tag{36}$$

Gradients of matrixes: I needed to use three different kinds of matrix gradients to duplicate the functionality of the Christoffel symbols, and there are three different notations for gradients in equation (36), so I extended the notations to matrixes in a way that seemed most consistent to me.

$$\frac{\partial \mathbf{S}}{\partial \mathbf{r}} = \begin{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{S}_{11}}{\partial x} & \frac{\partial \mathbf{S}_{12}}{\partial x} & \frac{\partial \mathbf{S}_{13}}{\partial x} \\ \frac{\partial \mathbf{S}_{21}}{\partial x} & \frac{\partial \mathbf{S}_{22}}{\partial x} & \frac{\partial \mathbf{S}_{23}}{\partial x} \\ \frac{\partial \mathbf{S}_{31}}{\partial x} & \frac{\partial \mathbf{S}_{32}}{\partial x} & \frac{\partial \mathbf{S}_{33}}{\partial x} \end{bmatrix} \\ \\ \begin{bmatrix} \frac{\partial \mathbf{S}_{11}}{\partial y} & \frac{\partial \mathbf{S}_{12}}{\partial y} & \frac{\partial \mathbf{S}_{13}}{\partial y} \\ \frac{\partial \mathbf{S}_{21}}{\partial y} & \frac{\partial \mathbf{S}_{22}}{\partial y} & \frac{\partial \mathbf{S}_{23}}{\partial y} \\ \frac{\partial \mathbf{S}_{31}}{\partial y} & \frac{\partial \mathbf{S}_{32}}{\partial y} & \frac{\partial \mathbf{S}_{33}}{\partial y} \end{bmatrix} \\ \\ \begin{bmatrix} \frac{\partial \mathbf{S}_{11}}{\partial z} & \frac{\partial \mathbf{S}_{12}}{\partial z} & \frac{\partial \mathbf{S}_{13}}{\partial z} \\ \frac{\partial \mathbf{S}_{21}}{\partial z} & \frac{\partial \mathbf{S}_{22}}{\partial z} & \frac{\partial \mathbf{S}_{23}}{\partial z} \\ \frac{\partial \mathbf{S}_{31}}{\partial z} & \frac{\partial \mathbf{S}_{32}}{\partial z} & \frac{\partial \mathbf{S}_{33}}{\partial z} \end{bmatrix} \end{bmatrix}, \quad \nabla_{\mathsf{T}}\mathbf{S} = \begin{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{S}_{11}}{\partial x} & \frac{\partial \mathbf{S}_{11}}{\partial y} & \frac{\partial \mathbf{S}_{11}}{\partial z} \\ \frac{\partial \mathbf{S}_{12}}{\partial x} & \frac{\partial \mathbf{S}_{12}}{\partial y} & \frac{\partial \mathbf{S}_{12}}{\partial z} \\ \frac{\partial \mathbf{S}_{13}}{\partial x} & \frac{\partial \mathbf{S}_{13}}{\partial y} & \frac{\partial \mathbf{S}_{13}}{\partial z} \end{bmatrix} \\ \\ \begin{bmatrix} \frac{\partial \mathbf{S}_{21}}{\partial x} & \frac{\partial \mathbf{S}_{21}}{\partial y} & \frac{\partial \mathbf{S}_{21}}{\partial z} \\ \frac{\partial \mathbf{S}_{22}}{\partial x} & \frac{\partial \mathbf{S}_{22}}{\partial y} & \frac{\partial \mathbf{S}_{22}}{\partial z} \\ \frac{\partial \mathbf{S}_{23}}{\partial x} & \frac{\partial \mathbf{S}_{23}}{\partial y} & \frac{\partial \mathbf{S}_{23}}{\partial z} \end{bmatrix} \\ \\ \begin{bmatrix} \frac{\partial \mathbf{S}_{31}}{\partial x} & \frac{\partial \mathbf{S}_{31}}{\partial y} & \frac{\partial \mathbf{S}_{31}}{\partial z} \\ \frac{\partial \mathbf{S}_{32}}{\partial x} & \frac{\partial \mathbf{S}_{32}}{\partial y} & \frac{\partial \mathbf{S}_{32}}{\partial z} \\ \frac{\partial \mathbf{S}_{33}}{\partial x} & \frac{\partial \mathbf{S}_{33}}{\partial y} & \frac{\partial \mathbf{S}_{33}}{\partial z} \end{bmatrix} \end{bmatrix}, \quad \nabla^{\mathsf{T}}\mathbf{S} = \begin{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{S}_{11}}{\partial x} & \frac{\partial \mathbf{S}_{21}}{\partial x} & \frac{\partial \mathbf{S}_{31}}{\partial x} \\ \frac{\partial \mathbf{S}_{11}}{\partial y} & \frac{\partial \mathbf{S}_{21}}{\partial y} & \frac{\partial \mathbf{S}_{31}}{\partial y} \\ \frac{\partial \mathbf{S}_{11}}{\partial z} & \frac{\partial \mathbf{S}_{21}}{\partial z} & \frac{\partial \mathbf{S}_{31}}{\partial z} \end{bmatrix} \\ \\ \begin{bmatrix} \frac{\partial \mathbf{S}_{12}}{\partial x} & \frac{\partial \mathbf{S}_{22}}{\partial x} & \frac{\partial \mathbf{S}_{32}}{\partial x} \\ \frac{\partial \mathbf{S}_{12}}{\partial y} & \frac{\partial \mathbf{S}_{22}}{\partial y} & \frac{\partial \mathbf{S}_{32}}{\partial y} \\ \frac{\partial \mathbf{S}_{12}}{\partial z} & \frac{\partial \mathbf{S}_{22}}{\partial z} & \frac{\partial \mathbf{S}_{32}}{\partial z} \end{bmatrix} \\ \\ \begin{bmatrix} \frac{\partial \mathbf{S}_{13}}{\partial x} & \frac{\partial \mathbf{S}_{23}}{\partial x} & \frac{\partial \mathbf{S}_{33}}{\partial x} \\ \frac{\partial \mathbf{S}_{13}}{\partial y} & \frac{\partial \mathbf{S}_{23}}{\partial y} & \frac{\partial \mathbf{S}_{33}}{\partial y} \\ \frac{\partial \mathbf{S}_{13}}{\partial z} & \frac{\partial \mathbf{S}_{23}}{\partial z} & \frac{\partial \mathbf{S}_{33}}{\partial z} \end{bmatrix} \end{bmatrix}. \tag{37}$$

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[1] Parker, D. B., "General Relativity in Absolute Space and Time", 2022, preprint, https://pgu.org
[2] `https://en.wikipedia.org/wiki/Block_matrix`