Consistent notation for common coordinate systems

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INTRODUCTION

When I work problems that involve multiple coordinate systems, notation can become a problem itself. This technical note describes my personal preference for notation. It covers the four coordinate systems I use most often: cartesian (x, y, z), cylindrical (r_c, θ_c, z_c), spherical (r_s, θ_s, ϕ_s), and oblate spheroidal (r_o, θ_o, ϕ_o). I also include tables for converting between cartesian and the other coordinate systems, including differentials and unit vectors.

A simple motivating example of an equation that uses multiple coordinate systems at the same time is a field $\mathbf{f}$ that has two constant components a and b, where a is in the direction of the spherical radius, and b is in the direction of the cylindrical radius:

$$\mathbf{f} = a\,\hat{\mathbf{r}}_s + b\,\hat{\mathbf{r}}_c. \tag{1}$$

Equation (1) is simpler and more intuitive than either the purely spherical equation:

$$\mathbf{f} = (a + b\cos(\phi_s))\,\hat{\mathbf{r}}_s - b\sin(\phi_s)\,\hat{\boldsymbol{\phi}}_s, \tag{2}$$

or the purely cylindrical equation:

$$\mathbf{f} = \left(\frac{a r_c}{\sqrt{r_c^2 + z_c^2}} + b\right)\hat{\mathbf{r}}_c + \frac{a z_c}{\sqrt{r_c^2 + z_c^2}}\,\hat{\mathbf{z}}_c. \tag{3}$$

For another motivating example, compare equation (27e) for $\cos(\phi_o)$ to equation (21b) for $\sin(\phi_s)$ and $\cos(\phi_s)$, and you can see that a more concise and potentially more productive equation for $\cos(\phi_o)$ is to mix oblate and spherical coordinates:

$$\cos(\phi_o) = \frac{r_o}{\sqrt{r_o^2 + a^2\sin(\phi_s)^2}}\cos(\phi_s) \tag{4}$$

Between the different coordinate systems, the z and θ coordinates are identical: $z = z_c$ and $\theta_c = \theta_s = \theta_o$. Even though they are identical, I still usually use subscripts to distinguish between them because there are use cases where the difference is important. For example, when converting between cartesian and cylindrical coordinates without using subscripts, you can get equations like:

$$0 = z - z \quad\Longrightarrow\quad 0 = 0, \tag{5}$$

when what you really want is

$$0 = z - z_c \quad\Longrightarrow\quad z = z_c. \tag{6}$$

I leave the cartesian coordinates unsubscripted because they are the coordinate system in which all the others are defined. However, if I ever needed to give the cartesian coordinates subscripts, I would probably use a for absolute: (x_a, y_a, z_a). If I am working with two coordinate systems with no conflicting coordinates, such as cartesian and spherical, then I usually leave all the subscripts off: (x, y, z) and (r, θ, ϕ).

To minimize my brain space devoted to coordinate systems, I tried to make them as consistent as possible.

- Consistent radial coordinates. All radial coordinates are labeled as r with a mnemonic subscript. The radial coordinates always appear as the first coordinate. All radial components have a standard range of $0 \le r < \infty$. However, it often happens that it is acceptable to allow $r < 0$.
- Consistent positive x axises. The positive x axises in the coordinate systems are $(x, 0, 0)$, $(r_c, 0, 0)$, $(r_s, 0, 0)$, and $(r_o, 0, 0)$.

- Consistent x-y planes. The equatorial planes in the coordinate systems are: $(x, y, 0)$, $(r_c, \theta_c, 0)$, $(r_s, \theta_s, 0)$, and $(r_o, \theta_o, 0)$.
- Consistent azimuthal coordinates. In the three coordinate systems with an azimuthal angle θ, it appears as the second coordinate, it is labeled as θ with a mnemonic subscript, and it is equal in all three coordinate systems: $\theta_c = \theta_s = \theta_o$.
- Consistent signs of elevations. Points above the equatorial plane always have a positive third coordinate, points below the equatorial plane always have a negative third component.
- Consistent elevation angles. In the two coordinate systems with an elevation angle ϕ, it appears as the third coordinate and it is labeled as ϕ with a mnemonic subscript, even though it has different values in the two coordinate systems.
- Consistent centered angular ranges. The azimuthal angles θ all have a standard range of $-\pi \le \theta < \pi$. The elevation angles ϕ all have a standard range of $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$. However, it is often acceptable to allow the angles to assume values outside these ranges.

Two slightly different forms of the arc tangent functions are used in the tables below. The first is the standard arc tangent function $\operatorname{atan}(u)$, where usually $u = \frac{y}{x}$. The domain of the function is $-\infty \le u \le \infty$ and the range of the function is $-\frac{\pi}{2} \le \operatorname{atan}(u) \le \frac{\pi}{2}$. The differential is:

$$\mathrm{d}\operatorname{atan}(u) = \frac{\mathrm{d}u}{1+u^2}. \tag{7}$$

The second arc tangent function is $\operatorname{atanxy}(x, y)$, which corrects the value of $\operatorname{atan}(\frac{y}{x})$ for the quadrant of x and y. The range is $-\pi \le \operatorname{atanxy}(x, y) < \pi$.

$$\text{for } x \ge 0: \qquad \operatorname{atanxy}(x, y) = \operatorname{atan}(\frac{y}{x}), \tag{8}$$

$$\text{for } x \le 0,\ y > 0: \qquad \operatorname{atanxy}(x, y) = \operatorname{atan}(\frac{y}{x}) + \pi, \tag{9}$$

$$\text{for } x \le 0,\ y \le 0: \qquad \operatorname{atanxy}(x, y) = \operatorname{atan}(\frac{y}{x}) - \pi. \tag{10}$$

The differential is:

$$\mathrm{d}\operatorname{atanxy}(x, y) = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2+y^2}. \tag{11}$$

CYLINDRICAL COORDINATES (r_c, θ_c, z_c)

(x, y, z) in terms of (r_c, θ_c, z_c):

$$x = r_c \cos(\theta_c), \tag{12a}$$

$$y = r_c \sin(\theta_c), \tag{12b}$$

$$z = z_c, \tag{12c}$$

$$\mathrm{d}x = \cos(\theta_c)\,\mathrm{d}r_c - r_c \sin(\theta_c)\,\mathrm{d}\theta_c, \tag{13a}$$

$$\mathrm{d}y = \sin(\theta_c)\,\mathrm{d}r_c + r_c \cos(\theta_c)\,\mathrm{d}\theta_c, \tag{13b}$$

$$\mathrm{d}z = \mathrm{d}z_c, \tag{13c}$$

$$\hat{\mathbf{x}} = \cos(\theta_c)\,\hat{\mathbf{r}}_c - \sin(\theta_c)\,\hat{\boldsymbol{\theta}}_c, \tag{14a}$$

$$\hat{\mathbf{y}} = \sin(\theta_c)\,\hat{\mathbf{r}}_c + \cos(\theta_c)\,\hat{\boldsymbol{\theta}}_c, \tag{14b}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}_c. \tag{14c}$$

(r_c, θ_c, z_c) in terms of (x, y, z):

$$r_c = \sqrt{x^2 + y^2}, \tag{15a}$$

$$\theta_c = \mathrm{atanxy}(x, y), \quad \sin(\theta_c) = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos(\theta_c) = \frac{x}{\sqrt{x^2 + y^2}}, \tag{15b}$$

$$z_c = z, \tag{15c}$$

$$\mathrm{d}r_c = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2 + y^2}}, \tag{16a}$$

$$\mathrm{d}\theta_c = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}, \tag{16b}$$

$$\mathrm{d}z_c = \mathrm{d}z, \tag{16c}$$

$$\hat{\mathbf{r}}_c = \cos(\theta_c)\,\hat{\mathbf{x}} + \sin(\theta_c)\,\hat{\mathbf{y}} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}, \tag{17a}$$

$$\hat{\boldsymbol{\theta}}_c = -\sin(\theta_c)\,\hat{\mathbf{x}} + \cos(\theta_c)\,\hat{\mathbf{y}} = \frac{-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}, \tag{17b}$$

$$\hat{\mathbf{z}}_c = \hat{\mathbf{z}}. \tag{17c}$$

SPHERICAL COORDINATES (r_s, θ_s, ϕ_s)

(x, y, z) in terms of (r_s, θ_s, ϕ_s):

$$x = r_s \cos(\theta_s) \cos(\phi_s), \tag{18a}$$

$$y = r_s \sin(\theta_s) \cos(\phi_s), \tag{18b}$$

$$z = r_s \sin(\phi_s), \tag{18c}$$

$$\mathrm{d}x = \cos(\theta_s)\cos(\phi_s)\,\mathrm{d}r_s - r_s \sin(\theta_s)\cos(\phi_s)\,\mathrm{d}\theta_s - r_s \cos(\theta_s)\sin(\phi_s)\,\mathrm{d}\phi_s, \tag{19a}$$

$$\mathrm{d}y = \sin(\theta_s)\cos(\phi_s)\,\mathrm{d}r_s + r_s \cos(\theta_s)\cos(\phi_s)\,\mathrm{d}\theta_s - r_s \sin(\theta_s)\sin(\phi_s)\,\mathrm{d}\phi_s, \tag{19b}$$

$$\mathrm{d}z = \sin(\phi_s)\,\mathrm{d}r_s + r_s \cos(\phi_s)\,\mathrm{d}\phi_s, \tag{19c}$$

$$\hat{\mathbf{x}} = \cos(\theta_s)\cos(\phi_s)\,\hat{\mathbf{r}}_s - \sin(\theta_s)\,\hat{\boldsymbol{\theta}}_s - \cos(\theta_s)\sin(\phi_s)\,\hat{\boldsymbol{\phi}}_s, \tag{20a}$$

$$\hat{\mathbf{y}} = \sin(\theta_s)\cos(\phi_s)\,\hat{\mathbf{r}}_s + \cos(\theta_s)\,\hat{\boldsymbol{\theta}}_s - \sin(\theta_s)\sin(\phi_s)\,\hat{\boldsymbol{\phi}}_s, \tag{20b}$$

$$\hat{\mathbf{z}} = \sin(\phi_s)\,\hat{\mathbf{r}}_s + \cos(\phi_s)\,\hat{\boldsymbol{\phi}}_s. \tag{20c}$$

(r_s, θ_s, ϕ_s) in terms of (x, y, z):

$$r_s = \sqrt{x^2 + y^2 + z^2}, \tag{21a}$$

$$\theta_s = \mathrm{atanxy}(x, y), \quad \sin(\theta_s) = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos(\theta_s) = \frac{x}{\sqrt{x^2 + y^2}}, \tag{21b}$$

$$\phi_s = \mathrm{atan}\left(\frac{z}{\sqrt{x^2 + y^2}}\right), \quad \sin(\phi_s) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos(\phi_s) = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \tag{21c}$$

$$\mathrm{d}r_s = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z}{\sqrt{x^2 + y^2 + z^2}}, \tag{22a}$$

$$\mathrm{d}\theta_s = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}, \tag{22b}$$

$$\mathrm{d}\phi_s = \frac{-zx\,\mathrm{d}x - zy\,\mathrm{d}y + (x^2 + y^2)\,\mathrm{d}z}{\sqrt{x^2 + y^2}(x^2 + y^2 + z^2)}, \tag{22c}$$

$$\hat{\mathbf{r}}_s = \cos(\theta_s)\cos(\phi_s)\,\hat{\mathbf{x}} + \sin(\theta_s)\cos(\phi_s)\,\hat{\mathbf{y}} + \sin(\phi_s)\,\hat{\mathbf{z}} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}, \tag{23a}$$

$$\hat{\boldsymbol{\theta}}_s = -\sin(\theta_s)\,\hat{\mathbf{x}} + \cos(\theta_s)\,\hat{\mathbf{y}} = \frac{-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}, \tag{23b}$$

$$\hat{\boldsymbol{\phi}}_s = -\cos(\theta_s)\sin(\phi_s)\,\hat{\mathbf{x}} - \sin(\theta_s)\sin(\phi_s)\,\hat{\mathbf{y}} + \cos(\phi_s)\,\hat{\mathbf{z}} = \frac{-xz\,\hat{\mathbf{x}} - yz\,\hat{\mathbf{y}} + (x^2 + y^2)\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}}. \tag{23c}$$

OBLATE SPHEROIDAL COORDINATES (r_o, θ_o, ϕ_o)

(x, y, z) in terms of (r_o, θ_o, ϕ_o):

$$x = \sqrt{r_o^2 + a^2}\cos(\theta_o)\cos(\phi_o), \tag{24a}$$

$$y = \sqrt{r_o^2 + a^2}\sin(\theta_o)\cos(\phi_o), \tag{24b}$$

$$z = r_o \sin(\phi_o), \tag{24c}$$

$$\mathrm{d}x = \frac{r_o}{\sqrt{r_o^2 + a^2}}\cos(\theta_o)\cos(\phi_o)\,\mathrm{d}r_o - \sqrt{r_o^2 + a^2}\sin(\theta_o)\cos(\phi_o)\,\mathrm{d}\theta_o - \sqrt{r_o^2 + a^2}\cos(\theta_o)\sin(\phi_o)\,\mathrm{d}\phi_o, \tag{25a}$$

$$\mathrm{d}y = \frac{r_o}{\sqrt{r_o^2 + a^2}}\sin(\theta_o)\cos(\phi_o)\,\mathrm{d}r_o + \sqrt{r_o^2 + a^2}\cos(\theta_o)\cos(\phi_o)\,\mathrm{d}\theta_o - \sqrt{r_o^2 + a^2}\sin(\theta_o)\sin(\phi_o)\,\mathrm{d}\phi_o, \tag{25b}$$

$$\mathrm{d}z = \sin(\phi_o)\,\mathrm{d}r_o + r_o\cos(\phi_o)\,\mathrm{d}\phi_o, \tag{25c}$$

$$\hat{\mathbf{x}} = \frac{r_o\cos(\theta_o)\cos(\phi_o)}{\sqrt{r_o^2 + a^2\sin(\phi_o)^2}}\,\hat{\mathbf{r}}_o - \sin(\theta_o)\,\hat{\boldsymbol{\theta}}_o - \frac{\sqrt{r_o^2 + a^2}\cos(\theta_o)\sin(\phi_o)}{\sqrt{r_o^2 + a^2\sin(\phi_o)^2}}\,\hat{\boldsymbol{\phi}}_o, \tag{26a}$$

$$\hat{\mathbf{y}} = \frac{r_o\sin(\theta_o)\cos(\phi_o)}{\sqrt{r_o^2 + a^2\sin(\phi_o)^2}}\,\hat{\mathbf{r}}_o + \cos(\theta_o)\,\hat{\boldsymbol{\theta}}_o - \frac{\sqrt{r_o^2 + a^2}\sin(\theta_o)\sin(\phi_o)}{\sqrt{r_o^2 + a^2\sin(\phi_o)^2}}\,\hat{\boldsymbol{\phi}}_o, \tag{26b}$$

$$\hat{\mathbf{z}} = \frac{\sqrt{r_o^2 + a^2}\sin(\phi_o)\,\hat{\mathbf{r}}_o + r_o\cos(\phi_o)\,\hat{\boldsymbol{\phi}}_o}{\sqrt{r_o^2 + a^2\sin(\phi_o)^2}}. \tag{26c}$$

(r_o, θ_o, ϕ_o) in terms of (x, y, z):

$$\text{implicit: } r_o^4 = (x^2 + y^2 + z^2 - a^2) r_o^2 + a^2 z^2, \tag{27a}$$

$$\text{explicit: } r_o = \sqrt{\frac{1}{2}(x^2 + y^2 + z^2 - a^2) + \sqrt{\frac{1}{4}(x^2 + y^2 + z^2 - a^2)^2 + a^2 z^2}}, \tag{27b}$$

$$\theta_o = \operatorname{atanxy}(x, y), \quad \sin(\theta_o) = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos(\theta_o) = \frac{x}{\sqrt{x^2 + y^2}}, \tag{27c}$$

$$\phi_o = \operatorname{atan}\left(\frac{\sqrt{r_o^2 + a^2}}{r_o} \frac{z}{\sqrt{x^2 + y^2}}\right), \quad \sin(\phi_o) = \frac{\sqrt{r_o^2 + a^2}}{\sqrt{r_o^2 + a^2 \frac{z^2}{x^2+y^2+z^2}}} \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \tag{27d}$$

$$\cos(\phi_o) = \frac{r_o}{\sqrt{r_o^2 + a^2 \frac{z^2}{x^2+y^2+z^2}}} \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}, \tag{27e}$$

$$\mathrm{d}r_o = \frac{r_o \sqrt{r_o^2 + a^2} \cos(\theta_o) \cos(\phi_o)\, \mathrm{d}x + r_o \sqrt{r_o^2 + a^2} \sin(\theta_o) \cos(\phi_o)\, \mathrm{d}y + (r_o^2 + a^2) \sin(\phi_o)\, \mathrm{d}z}{r_o^2 + a^2 \sin(\phi_o)^2}, \tag{28a}$$

$$\mathrm{d}\theta_o = \frac{-\sin(\theta_o)\, \mathrm{d}x + \cos(\theta_o)\, \mathrm{d}y}{\sqrt{r_o^2 + a^2} \cos(\phi_o)}, \tag{28b}$$

$$\mathrm{d}\phi_o = \frac{-\sqrt{r_o^2 + a^2} \cos(\theta_o) \sin(\phi_o)\, \mathrm{d}x - \sqrt{r_o^2 + a^2} \sin(\theta_o) \sin(\phi_o)\, \mathrm{d}y + r_o \cos(\phi_o)\, \mathrm{d}z}{r_o^2 + a^2 \sin(\phi_o)^2}, \tag{28c}$$

$$\hat{\mathbf{r}}_o = \frac{r_o \cos(\theta_o) \cos(\phi_o)\, \hat{\mathbf{x}} + r_o \sin(\theta_o) \cos(\phi_o)\, \hat{\mathbf{y}} + \sqrt{r_o^2 + a^2} \sin(\phi_o)\, \hat{\mathbf{z}}}{\sqrt{r_o^2 + a^2 \sin(\phi_o)^2}}, \tag{29a}$$

$$\hat{\boldsymbol{\theta}}_o = -\sin(\theta_o)\, \hat{\mathbf{x}} + \cos(\theta_o)\, \hat{\mathbf{y}}, \tag{29b}$$

$$\hat{\boldsymbol{\phi}}_o = \frac{-\sqrt{r_o^2 + a^2} \cos(\theta_o) \sin(\phi_o)\, \hat{\mathbf{x}} - \sqrt{r_o^2 + a^2} \sin(\theta_o) \sin(\phi_o)\, \hat{\mathbf{y}} + r_o \cos(\phi_o)\, \hat{\mathbf{z}}}{\sqrt{r_o^2 + a^2 \sin(\phi_o)^2}}. \tag{29c}$$

Notes on oblate spheroidal coordinates:

1. The parameter a defines how oblate the spheriod is. For $a = 0$, oblate spheroidal coordinates become spherical coordinates.
2. The oblate spheroidal coordinates are messy (especially r_o), so I do not expand them out when writing (r_o, θ_o, ϕ_o) in terms of (x, y, z). When I experimented with expanding them out, I was not enlightened by the results.
3. To derive the implicit formula for r_o, equation (27a), start by squaring and adding equations (24a) through (24c):

$$x^2 + y^2 + z^2 = (r_o^2 + a^2) \cos(\theta_o)^2 \cos(\phi_o)^2 + (r_o^2 + a^2) \sin(\theta_o)^2 \cos(\phi_o)^2 + r_o^2 \sin(\phi_o)^2 \tag{30a}$$

$$= r_o^2 + a^2 \cos(\phi_o)^2 \tag{30b}$$

$$= r_o^2 + a^2 (1 - \sin(\phi_o)^2) \tag{30c}$$

$$= r_o^2 + a^2 (1 - \frac{z^2}{r_o^2}) \qquad \text{(squaring equation (24c) and solving for } \sin(\phi_o)^2\text{)} \tag{30d}$$

   If we now multiply both sides by r_o^2 and rearrange terms, we get equation (27a).
4. To derive the explicit formula for r_o, equation (27b), solve equation (27a) as a quadratic equation in r_o^2, take the positive root, and then solve that as a quadratic equation in r_o.
5. For converting oblate spheroidal coordinates to and from cartesian, I found the method described in [1] to be the easiest, especially when implemented using a computer algebra system.

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[1] Parker, D.B., "Implicit coordinate system transforms", 2023, preprint, `https://pgu.org`