General Relativity with Electromagnetism in Absolute Space and Time

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(Dated: September 11, 2022)

This letter describes how to implement general relativity with electromagnetism in absolute space and time, so that a simulation of both together takes only on the order of twice as much time as a simulation of electromagnetism alone. The implementation is eminently suited for parallel computation. This allows faster and algorithmically simpler simulations of relativistic charged particles propagating through intense electromagetic fields near relativistic bodies. Maxwell's equations are already written in terms of absolute space and time, so almost all of the speed improvements and algorithmic simplifications come from specializing general relativity to absolute space and time.

INTRODUCTION

Accurately simulating electrodynamics is efficient and straightforward because Maxwell's equation are already written in terms of absolute space and time, so electrodynamics simulations can take uniform steps through space and time. Accurately simulating general relativity is trickier because Einstein's equations usually necessitate nonuniform step sizes through space and time. Reconciling the differing step sizes for electromagetism and general relativity leads to algorithmic complexity and computational inefficiency. This letter describes how to implement general relativity with electromagnetism in absolute space and time, so that a simulation of both together takes only on the order of twice as much time as a simulation of electromagnetism alone. This allows faster and algorithmically simpler simulations of highly relativistic charged particles propagating through intense electromagetic fields near relativistic bodies.

Here is the most important result in this letter, the 3-dimensional force equation for absolute electrogravity:

$$\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2} = -\left(g^{i\sigma} - \frac{1}{c} g^{0\sigma} \frac{\mathrm{d}r^i}{\mathrm{d}t}\right) [\mu\nu, \sigma] \frac{\mathrm{d}r^\mu}{\mathrm{d}t} \frac{\mathrm{d}r^\nu}{\mathrm{d}t} + \frac{kq}{m} \sqrt{g_{00}} \left(cF^i_{\ \nu} - F^0_{\ \nu} \frac{\mathrm{d}r^i}{\mathrm{d}t}\right) \frac{\mathrm{d}r^\nu}{\mathrm{d}t}. \tag{1}$$

Each of the two terms on the right hand side, one for general relativity and the other for electromagnetism, take roughly the same amount of time to compute (depending on the hardware and programming tricks) so the total computation time is on the order of twice as much as either term alone. In addition, equation (1) is eminently suited to parallel computation because it allows space to be divided up into independent rectangular blocks.

There are other ways to simulate general relativity and electromagnetism other than by merely stepping uniformly through space and time (such as adaptive meshes, etc), but I expect those other methods will also benefit from equation (1).

Anyone sufficiently familiar with general relativity might want to skip directly to equation (9) in the derivation because the material from here to there is mainly review.

Equation (1) is written using Einstein's summation convention. The indexes μ, ν, and σ run from 0 to 3. The index i runs from 1 to 3. Equation (1) represents a system of three equations for $i = 1, 2, 3$. t is absolute time. The r^i are the absolute coordinates (r_x, r_y, r_z) of a particle in an absolute (x, y, z) coordinate system, so that $\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2}$ is the absolute acceleration of a particle that is moving at absolute velocity $\frac{\mathrm{d}r^i}{\mathrm{d}t}$. The $[\mu\nu, \sigma]$ are Christoffel symbols of the first kind. Inside the Christoffel symbols, it is important to remember that $x^0 = ct$ and $x^1, x^2, x^3 = x, y, z$. In the summations on the right hand side, it is important to remember that $r^0 = ct$, so that $\frac{\mathrm{d}r^0}{\mathrm{d}t} = c$. The units for t and x, y, z are ordinary time and length units; for example, seconds and meters. k is the electromagnetic constant, q is the charge, m is the mass, and F is the electromagnetic field tensor. No constants have been set to 1.

In absolute electrogravity, the field equation for the generation and propagation of $g_{\mu\nu}$ is the same as the Einstein equation in general relativity, as long as one remembers to include electromagnetism in the energy-momentum-stress tensor. Also, the step sizes when propagating $g_{\mu\nu}$ should be the same size as in the simulation of the force law. In absolute electrogravity, $g_{\mu\nu}$ is no longer a metric; it is a set of 10 potentials (taking advantage of the symmetry of $g_{\mu\nu}$). The field equation for the propagation of F is the same as Maxwell's equations.

This work is a continuation of [2]. I will be using equation (1) in my own simulations. I hope others find it as useful.

DERIVING THE FORCE EQUATION

We assume that the metric $g_{\alpha\beta}$ has signature $(1,-1,-1,-1)$ so that τ and $x^0 = ct$ are both positive in the future direction.

We start from the geodesic equation including the electromagnetic force:

$$\frac{\mathrm{d}^2 r^\alpha}{\mathrm{d}\tau^2} = -\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}r^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}r^\nu}{\mathrm{d}\tau} + \frac{kq}{m}F^\alpha_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}\tau}. \tag{2}$$

τ is the proper time as measured by a standard clock moving with the particle. The trajectory of the particle in four dimensions is $r^\alpha = (r^0, r^1, r^2, r^3)$ where (r^0, r^1, r^2, r^3) are the coordinates of the particle in the coordinate system $x^\alpha = (x^0, x^1, x^2, x^3)$. x^0 is taken to be time coordinate $x^0 = ct$, where in general relativity x^0 and/or t are coordinate time. k is the electromagnetic constant, q is the charge, m is the mass, and F is the electromagnetic field tensor.

Later we will take $r^0 = x^0 = ct$, with t the absolute time, and (x^1, x^2, x^3) to be the absolute (x, y, z) coordinate system, but there is work to be done before we can make that leap.

The Christoffel symbol of the second kind in equation (2), $\Gamma^\alpha_{\mu\nu}$, conflates the role of the inverse metric, $g^{\alpha\beta}$ with the derivatives of the metric $g_{\alpha\beta}$ in calculating the geodesic:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{3}$$

We are going to revert to a somewhat more archaic notation (see [1], equations (20d) through (23)) using Christoffel symbols of the first kind, written as $[\mu\nu, \sigma]$, to separate the inverse metric from the derivatives of the metric:

$$[\mu\nu, \sigma] = \frac{1}{2}\left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{4}$$

Substituting equation (4) into equation (3) for $\Gamma^\alpha_{\mu\nu}$ gives:

$$\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}[\mu\nu, \sigma]. \tag{5}$$

Substituting equation (5) into equation (2) for the geodesic gives:

$$\frac{\mathrm{d}^2 r^\alpha}{\mathrm{d}\tau^2} = -g^{\alpha\sigma}[\mu\nu, \sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}r^\nu}{\mathrm{d}\tau} + \frac{kq}{m}F^\alpha_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}\tau}. \tag{6}$$

We now want to get rid of the derivatives with respect to τ (the proper time), and replace them with derivatives with respect to x^0 (the coordinate time). Applying the chain rule for derivatives gives:

$$\frac{\mathrm{d}r^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}r^\alpha}{\mathrm{d}x^0}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}, \qquad \frac{\mathrm{d}^2 r^\alpha}{\mathrm{d}\tau^2} = \left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}^2 r^\alpha}{\mathrm{d}x^{0\,2}} + \frac{\mathrm{d}r^\alpha}{\mathrm{d}x^0}\frac{\mathrm{d}^2 x^0}{\mathrm{d}\tau^2}. \tag{7}$$

Substituting the derivatives from equations (7) into equation (6):

$$\frac{\mathrm{d}^2 r^\alpha}{\mathrm{d}x^{0\,2}}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}r^\alpha}{\mathrm{d}x^0}\frac{\mathrm{d}^2 x^0}{\mathrm{d}\tau^2} = -g^{\alpha\sigma}[\mu\nu, \sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2 + \frac{kq}{m}F^\alpha_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}. \tag{8}$$

Multiplying both sides of equation (8) by $\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-2}$:

$$\frac{\mathrm{d}^2 r^\alpha}{\mathrm{d}x^{0\,2}} + \frac{\mathrm{d}r^\alpha}{\mathrm{d}x^0}\left\{\frac{\mathrm{d}^2 x^0}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-2}\right\} = -g^{\alpha\sigma}[\mu\nu, \sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0} + \frac{kq}{m}F^\alpha_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-1}. \tag{9}$$

We are entering the crucial steps where we specialize general relativity to absolute space and time. The first step is to rewrite the factor $\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-1}$ on the right hand side of equation (9):

$$\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-1} = \frac{\mathrm{d}\tau}{\mathrm{d}x^0} = \sqrt{\frac{(\mathrm{d}\tau)^2}{(\mathrm{d}x^0)^2}} = \sqrt{\frac{g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}{(\mathrm{d}x^0)^2}} = \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0}}. \tag{10}$$

We now specialize space to be absolute so that we can run simulations with uniform distance steps. In absolute space the coordinate system does not change with time, so the derivatives $\frac{\mathrm{d}x^\nu}{\mathrm{d}x^0}$ in equation (10) are:

$$\frac{\mathrm{d}x^0}{\mathrm{d}x^0} = 1, \qquad \frac{\mathrm{d}x^1}{\mathrm{d}x^0} = 0, \qquad \frac{\mathrm{d}x^2}{\mathrm{d}x^0} = 0, \qquad \frac{\mathrm{d}x^3}{\mathrm{d}x^0} = 0. \tag{11}$$

Substituting the derivatives from equations (11) into equation (10):

$$\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-1} = \sqrt{g_{00}}. \tag{12}$$

Substituting equation (12) into equation (9):

$$\frac{\mathrm{d}^2 r^\alpha}{\mathrm{d}{x^0}^2} + \frac{\mathrm{d}r^\alpha}{\mathrm{d}x^0}\left\{\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-2}\right\} = -g^{\alpha\sigma}[\mu\nu,\sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0} + \frac{kq}{m}\sqrt{g_{00}}F^\alpha_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}. \tag{13}$$

The left hand side of equation (13) still has a complicated factor in curly braces. Actually, equation (13) represents four equations, one for each of $\alpha = 0, 1, 2, 3$. The complicated factor, because it depends only on x^0 and τ, is the same for all four equations. We are going to sacrifice one of the equations to solve for the complicated factor. In particular, we are going to sacrifice the equation for $\alpha = 0$.

Separating equation (9) into equations for $\alpha = 0$ and $\alpha = 1, 2, 3$, and then using i instead of α to remind ourselves that the index in equation (14b) now only goes from 1 to 3:

$$\frac{\mathrm{d}^2 r^0}{\mathrm{d}{x^0}^2} + \frac{\mathrm{d}r^0}{\mathrm{d}x^0}\left\{\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-2}\right\} = -g^{0\sigma}[\mu\nu,\sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0} + \frac{kq}{m}\sqrt{g_{00}}F^0_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}, \tag{14a}$$

$$\frac{\mathrm{d}^2 r^i}{\mathrm{d}{x^0}^2} + \frac{\mathrm{d}r^i}{\mathrm{d}x^0}\left\{\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-2}\right\} = -g^{i\sigma}[\mu\nu,\sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0} + \frac{kq}{m}\sqrt{g_{00}}F^i_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}. \tag{14b}$$

We now specialize time to be absolute so that we can run simulations with uniform time steps. In absolute time, $r^0 = x^0$. Then the derivatives of r^0 on the left hand side of equation (14a) become:

$$\frac{\mathrm{d}r^0}{\mathrm{d}x^0} = 1, \qquad \frac{\mathrm{d}^2r^0}{\mathrm{d}{x^0}^2} = 0. \tag{15}$$

Solving for the complicated factor by substituting equations (15) into equation (14a) gives:

$$\left\{\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^{-2}\right\} = -g^{0\sigma}[\mu\nu,\sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0} + \frac{kq}{m}\sqrt{g_{00}}F^0_\nu\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}. \tag{16}$$

Substituting equation (16) into equation (14b) and moving the term containing the complicated factor to the right hand side gives:

$$\frac{\mathrm{d}^2 r^i}{\mathrm{d}{x^0}^2} = -\left(g^{i\sigma} - g^{0\sigma}\frac{\mathrm{d}r^i}{\mathrm{d}x^0}\right)[\mu\nu,\sigma]\frac{\mathrm{d}r^\mu}{\mathrm{d}x^0}\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0} + \frac{kq}{m}\sqrt{g_{00}}\left(F^i_\nu - F^0_\nu\frac{\mathrm{d}r^i}{\mathrm{d}x^0}\right)\frac{\mathrm{d}r^\nu}{\mathrm{d}x^0}. \tag{17}$$

Finally, replacing x^0 with ct and multiplying both sides of equation (17) by c^2 gives equation (1).

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[1] Einstein, A., "The Foundation of the General Theory of Relativity", 1916, pgs 111-164, translated by Perrett, W., and Jeffery G. B., *The Principle of Relativity*, Dover, 1952
[2] Parker, D. B., "General Relativity in Absolute Space and Time", preprint, https://pgu.org